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A Lipschitz condition for the width function of convex bodies in arbitrary Minkowski spaces

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ABSTRACT

Studying first the Euclidean subcase, we show that the Minkowskian width function of a convex body in an n-dimensional (normed linear or) Minkowski space satisfies a specified Lipschitz condition.

1. Introduction

The study of width functions of convex bodies was already stimulated in the classical monograph [3] (see Section 33 there). These functions play an important role in the fields of geometric convexity, geometric tomography, geometric inequalities, and Minkowski geometry; cf. [12,6,4,13], respectively. More precisely, width functions of convex bodies are basic for the following topics and notions from these fields: *support functions* of convex bodies (see [12], Section 1.7), the *difference body* and the *central symmetral* of a convex body (and therefore also the related *maximum chord-length function*; cf. [6], Section 3.2 and [1]), *bodies of constant width* (see the surveys [5,8,10]) and the related class of *reduced bodies* [7,9,2], *diameter* and *thickness* as extremal values of width functions (leading to famous topics like the isodiametric problem, or the theorems of Jung and Steinhagen; cf. [3], Section 44, [4], Section 11, and [11]), and problems involving the *mean width* of convex bodies (see again [4], Section 11).

2. Results and proofs

In what follows, let K denote a *convex body* in $\mathbb{R}^n$ for some $n \geq 2$, i.e., a compact, convex set whose affine hull aff(K) equals $\mathbb{R}^n$. The n-dimensional *Euclidean unit ball* is denoted by $E = E_n$. Hence, if $\langle \cdot, \cdot \rangle$ is the standard scalar product in $\mathbb{R}^n$, one has

$$E_n = \{v \in \mathbb{R}^n | \langle v, v \rangle \leq 1\}.$$

Moreover, we put, as usual, $S^{n-1} := \partial E_n$.

Let B denote the unit ball of an arbitrary (normed linear or) Minkowski space on $\mathbb{R}^n$, i.e., B is a convex body in $\mathbb{R}^n$ centered at the origin. Thus the induced *Minkowskian norm* $\|\cdot\|_B$ satisfies

$$B = \{v \in \mathbb{R}^n : \|v\|_B \leq 1\}.$$

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For $u \in S^{n-1}$, let $H(K, u)$ denote the *supporting hyperplane* of K with outward normal vector u in the Euclidean sense.

The *Minkowskian width function* $w_B(K, \cdot) : S^{n-1} \to \mathbb{R}^+$ is defined by

$$w_B(K, u) := \min\{\|x - y\|_B : x \in H(K, u), y \in H(K, -u)\}. \tag{1}$$

This means: $w_B(K, u)$ is the Minkowskian distance between $H(K, u)$ and $H(K, -u)$. To prove that $w_B(K, \cdot)$ satisfies a specified Lipschitz condition, we study first the Euclidean case $B = E = E_n$. The Euclidean norm is denoted by $\| \cdot \|_E$. For brevity, we write

$$w(u) := w_E(K, u) \quad \text{for } u \in S^{n-1}. \tag{2}$$

Furthermore, the *diameter* diam K and the *thickness* $\Delta(K)$ in the Euclidean sense are defined by

$$\operatorname{diam} K := \max_{x,y \in K} \|x - y\|_E = \max_{u \in S^{n-1}} w(u) \quad \text{and} \tag{3}$$

$$\Delta(K) := \min_{u \in S^{n-1}} w(u), \tag{4}$$

respectively.

As announced, we start with the Euclidean subcase.

Proposition. *For all $u, v \in S^{n-1}$, the inequality*

$$|w(v) - w(u)| \le \operatorname{diam} K \cdot \|v - u\|_E \tag{5}$$

holds.

Proof. We may assume that $u \neq v$. In the case $\frac{\pi}{2} < \sphericalangle(u, v) \le \pi$ one has $\|v - u\|_E \ge \|v + u\|_E$. Since $w(u) = w(-u)$, we can therefore also suppose that $\alpha := \sphericalangle(u, v) \le \frac{\pi}{2}$, and hence $\langle u, v \rangle = \cos\alpha \ge 0$.

Put

$$\begin{aligned}
&H_1 := H(K, u), \qquad H_1' := H(K, -u),\\
&H_2 := H(K, v), \qquad H_2' := H(K, -v);\\
&z := \frac{1}{\|v - \langle v, u\rangle \cdot u\|_E} \cdot (v - \langle u, v\rangle \cdot u) \in S^{n-1},\\
&H_0 := H(K, z), \qquad H_0' := H(K, -z).
\end{aligned}$$

Moreover, let $P_0 \subset \mathbb{R}^n$ denote the – homogeneous – plane spanned by the unit vectors u and v. Without loss of generality, we may suppose that

$$F := K \cap P_0 \neq \emptyset.$$

Furthermore, put

$$L_i := H_i \cap P_0, \qquad L_i' := H_i' \cap P_0 \quad \text{for } 0 \le i \le 2.$$

Then all L_i, L_i' are – affine – lines in P_0, and F is contained in the two-dimensional strips $\operatorname{conv}(L_i \cup L_i')$ for $0 \le i \le 2$, where conv denotes convex hull.

Note that F does not necessarily touch the lines L_i, L_i'. We merely know that K touches all six hyperplanes H_i, H_i' for $0 \le i \le 2$. Since $\langle u, z\rangle = 0$, the following holds: The lines L_0, L_0' are parallel to the homogeneous line $\mathbb{R} \cdot u$, while the lines L_1, L_1' are parallel to the homogeneous line $\mathbb{R} \cdot z$. Hence, the four points $a_1, a_2, a_3, a_4 \in P_0$ given by

$$\begin{aligned}
&\{a_1\} = L_0' \cap L_1', \qquad \{a_2\} = L_0' \cap L_1,\\
&\{a_3\} = L_0 \cap L_1, \qquad \{a_4\} = L_0 \cap L_1'
\end{aligned}$$

are the vertices of a rectangle (Fig. 1). Without loss of generality, we may assume that

$$a_1 = 0, \qquad a_2 = d \cdot u, \qquad a_3 = d \cdot u + h \cdot z, \qquad a_4 = h \cdot z,$$

where $d := w(u)$ and $h := w(z)$.

Note that, for $0 \le i \le 2$, L_i and L_i' have the same Euclidean distance as H_i and H_i', because $\{u, v, z\} \subseteq P_0$.

Let H_3 or H_3' denote the hyperplanes in $\mathbb{R}^n$ that are parallel to $H_2 = H(K, v)$ and pass through a_1 or a_3, respectively. Then one has

$$K \subseteq \operatorname{conv}(H_0 \cup H_0') \cap \operatorname{conv}(H_1 \cup H_1') \subseteq \operatorname{conv}(H_3 \cup H_3')$$

and, hence,

$$w(v) \le \langle a_3, v\rangle.$$

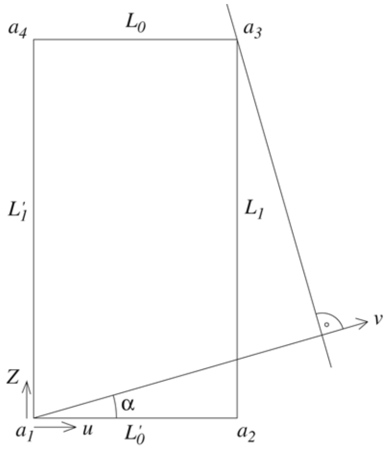

Fig. 1.

Since $0 < \alpha \le \frac{\pi}{2}$, we have

$$v = \cos\alpha \cdot u + \sin\alpha \cdot z.$$

Therefore we get

$$\|v-u\|_E = \sqrt{(1-\cos\alpha)^2 + \sin^2\alpha} = \sqrt{2 - 2\cdot\cos\alpha},$$
$$w(v) - w(u) \le \cos\alpha \cdot d + \sin\alpha \cdot h - d = \sin\alpha \cdot h - (1-\cos\alpha)\cdot d.$$

This implies

$$\begin{aligned}\frac{w(v)-w(u)}{\|v-u\|_E} &< h \cdot \frac{\sin\alpha}{\sqrt{2-2\cdot\cos\alpha}} \\ &= h \cdot \sqrt{\frac{1-\cos^2\alpha}{2\cdot(1-\cos\alpha)}} \\ &= h \cdot \sqrt{\frac{1}{2}\cdot(1+\cos\alpha)} \\ &\le h \le \operatorname{diam} K.\end{aligned}$$

On exchanging the roles of u and v, (5) follows. □

Remarks. (i) As pointed out to us by Rolf Schneider, Lemma 1.8.10 in [12] implies the following, slightly weaker Lipschitz condition:

$$|w(v) - w(u)| \le 2 \cdot R \cdot \|v-u\|_E. \tag{6}$$

Here R denotes the circumradius of K; that is the radius of the uniquely determined smallest Euclidean ball containing K.

(ii) The estimate (5) is sharp in the following sense: For every $\eta > 0$, there exist a compact and convex body K as well as $u, v \in S^{n-1}$ satisfying

$$|w(v) - w(u)| > (1-\eta)\cdot \operatorname{diam} K \cdot \|v-u\|_E. \tag{7}$$

Namely, let $K \subseteq \mathbb{R}^2$ denote the rectangle with vertices

$$(0,0),\ (d,0),\ (d,h),\ (0,h),$$

where $0 < d < h$.

If $u = (1,0)$, then we get, similarly to in the above proof,

$$\begin{aligned}\lim_{v\to u, v\in S^{n-1}\setminus\{u\}} \frac{|w(v)-w(u)|}{\|v-u\|_E} &= \lim_{\alpha\to 0,\alpha>0} \frac{|\sin\alpha\cdot h - (1-\cos\alpha)\cdot d|}{\sqrt{2-2\cdot\cos\alpha}} = \lim_{\alpha\to 0,\alpha>0}\left(h\cdot\frac{\sin\alpha}{\sqrt{2-2\cdot\cos\alpha}}\right) \\ &= h\cdot\lim_{\alpha\to 0}\sqrt{\frac{1}{2}\cdot(1+\cos\alpha)} = h.\end{aligned}$$

Hence, if $\frac{h}{d}$ is so large that

$$h > (1-\eta)\cdot\sqrt{h^2+d^2} = (1-\eta)\cdot\operatorname{diam} K,$$

then (7) holds for $u = (1, 0)$ and $v = (\cos\alpha, \sin\alpha)$ if $\alpha \in \mathbb{R}^+$ is small enough. □

Now we return to arbitrary Minkowskian norms $\|\cdot\|_B$. Recall that all $u \in S^{n-1}$ satisfy

$$w_B(K, u) = 2 \cdot \frac{w_E(K, u)}{w_E(B, u)}. \tag{8}$$

See, for instance, [1,2]. On the basis of our Proposition and (8), we can now also prove the following:

Theorem. *For every convex body K in $\mathbb{R}^n$, $n \geq 2$, and every Minkowskian norm $\|\cdot\|_B$ on $\mathbb{R}^n$ one has*

$$\begin{aligned} |w_B(K, v) - w_B(K, u)| &\leq 2 \cdot \Delta(B)^{-2} \cdot \operatorname{diam} K \cdot (\Delta(B) + \operatorname{diam} B) \cdot \|v - u\|_E \\ &\leq 4 \cdot \Delta(B)^{-2} \cdot \operatorname{diam} B \cdot \operatorname{diam} K \cdot \|v - u\|_E \end{aligned} \tag{9}$$

for all $u, v \in S^{n-1}$.

Proof. The second estimate in (9) is trivial, because $\Delta(B) \leq \operatorname{diam} B$. Now assume that $u, v \in S^{n-1}$ are fixed. Our proposition, applied to the convex bodies K and B, yields

$$|w_E(K, v) - w_E(K, u)| \leq \operatorname{diam} K \cdot \|v - u\|_E,$$
$$|w_E(B, v) - w_E(B, u)| \leq \operatorname{diam} B \cdot \|v - u\|_E.$$

Combining this with (8), (3) and (4) we obtain

$$\begin{aligned} |w_B(K, v) - w_B(K, u)| &= 2 \cdot \left| \frac{w_E(K, v)}{w_E(B, v)} - \frac{w_E(K, u)}{w_E(B, u)} \right| \\ &= 2 \cdot \left| \frac{w_E(K, v) - w_E(K, u)}{w_E(B, v)} + w_E(K, u) \cdot \frac{w_E(B, u) - w_E(B, v)}{w_E(B, v) \cdot w_E(B, u)} \right| \\ &\leq 2 \cdot \left(\frac{|w_E(K, v) - w_E(K, u)|}{w_E(B, v)} + w_E(K, u) \cdot \frac{|w_E(B, u) - w_E(B, v)|}{w_E(B, v) \cdot w_E(B, u)} \right) \\ &\leq 2 \cdot \Big(\Delta(B)^{-1} \cdot \operatorname{diam} K + \Delta(B)^{-2} \cdot \operatorname{diam} K \cdot \operatorname{diam} B \Big) \cdot \|v - u\|_E \\ &= 2 \cdot \Delta(B)^{-2} \cdot \operatorname{diam} K \cdot (\Delta(B) + \operatorname{diam} B) \cdot \|v - u\|_E. \quad \square \end{aligned}$$

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